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On stellar proper motion and its updating

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ABSTRACT. Vector algebra and "local" coordinate systems on the celestial sphere are used to derive exact equations for computing the new mean place of a star after motion of the star. New rigorous matrix expressions are given for transforming proper motion components and radial velocities from one epoch of observation to another, taking into consideration the corresponding changes in epoch of orientation.

1. Introduction

Given α_0 and δ_0, the right ascension and declination of a star at time t_0, the associated unit vector of direction cosines pointing to the star, in a certain inertial (X,Y,Z) reference frame at epoch t_0, is given by

$$\hat{X}_0 = \begin{Bmatrix} \cos\delta_0 \cos\alpha_0 \\ \cos\delta_0 \sin\alpha_0 \\ \sin\delta_0 \end{Bmatrix} . \tag{1.1}$$

The first problem considered here is simply that of computing the vector of direction cosines $\hat{X}_1$, at another time t_1, pointing to the new position of the star after it has moved with constant velocity along a straight line in space from point A at t_0 to point B at t_1. Figure 1 shows the basic triangle formed by A, B, and the observer O, and labels various vectors used later. If needed, α_1 and δ_1 are recoverable from $\hat{X}_1$. For the purpose of this work, the observer is assumed fixed with respect to the baricenter of the solar system. That is, only the contribution of the motion of the star to the mean place is considered, excluding the contribution of parallax and aberration. The curvilinear components of velocity at time t_0, μ_{α_0} and μ_{δ_0} are assumed known. Exact star position reductions also require the radial velocity V and distance r (actually only $\nu = V/r$ is needed), in order to completely specify the orientation of the star's path in space. As is ordinarily necessary, r will be expressed in terms of the annual parallax Π.

Schlesinger [1917] was the first to discuss the importance of radial velocities in the reduction of star positions. Many classical textbooks on spherical astronomy after him neglect V and Π in corrections for proper motion. Smart [1956, p.249], the paper by Kustaanheimo [1960] and Woolard and Clemence [1966, p.307] all include approximate formulas in slightly different forms to obtain the first-time derivatives of the components of proper motion, incorporating radial velocity effects. Scott and Hughes [1964] implemented these approximations in their second-order matrix expansion of $\hat{X}_1$ in t ($=t_1-t_0$). A somewhat different matrix formulation using basically the same assumptions was presented by Mueller [1969, p.114] and Murray [1983, p.54]. In [Eichhorn and Rust, 1970], [Eichhorn, 1974, p.22], [Taff, 1981, p.37] and [Eichhorn, 1982] a rigorous algebraic approach based on the time derivatives of the star's original position vector is followed. Recent editions of the Astronomical Almanac [e.g., Astronomical Almanac, 1984, p.B36] give without proof or reference, a succint statement of an exact vector algorithm for computing $\hat{X}_1$. Green [1985, p.265] arrives at similar results based on the numerical normalization of a vector. Finally Stumpff [1985] expanded Schlesinger's ideas and incorporated special relativistic effects. None of the above authors except Eichhorn and Rust [1970] and later Eichhorn [1974, p.84] included equations for transforming proper motion components between epochs.

An alternative geometric approach exclusively dependent on the notion of rotation matrices and "local" coordinate frames is presented here. The equivalence with some of the conventional proper motion equations is then established.

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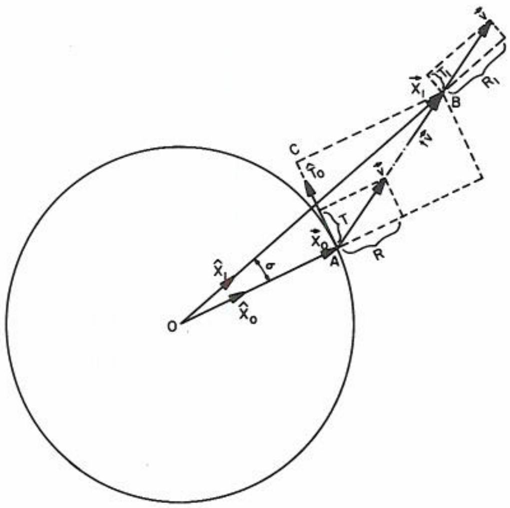

Fig. 1 - Vector relationships on the observer and velocity $\vec{v}$ plane.

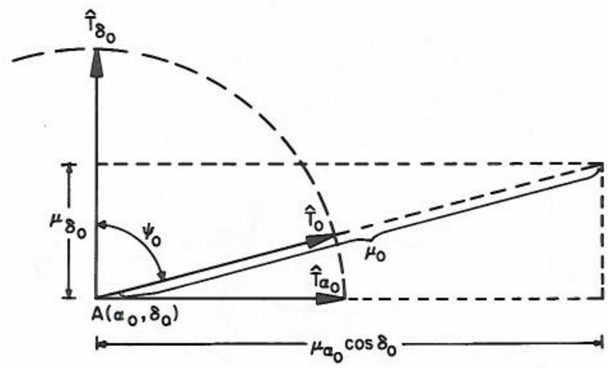

Fig. 2 - Proper motion components in right ascension and declination.

2. The problem of proper motion

It is seen from Fig. 1 that the position vector of a star at time t_1 can be written

$$\vec{X}_1 = \vec{X}_0 + t\vec{v} \; . \tag{2.1}$$

Introduce the unit vectors $\hat{X}_0$ and $\hat{X}_1$ along the directions $\vec{X}_0$ and $\vec{X}_1$ respectively, then since in general

$$\hat{X} = \vec{X}/|\vec{X}| \tag{2.2}$$

where

$$|\vec{X}| = (\vec{X} \cdot \vec{X})^{\frac{1}{2}} \; , \tag{2.3}$$

so that,

$$\hat{X}_0 = \vec{X}_0/r_0 \tag{2.4}$$

and

$$\hat{X}_1 = \vec{X}_1/r_1 \, , \tag{2.5}$$

where r_0 and r_1 denote $|\vec{X}_0|$ and $|\vec{X}_1|$ respectively.

The unit vector $\hat{T}_0$ tangent to the projection of the star path on the celestial sphere can be expressed as follows (see Fig. 2)

$$\hat{T}_0 = \sin\psi_0 \; \hat{T}_{\alpha_0} + \cos\psi_0 \; \hat{T}_{\delta_0} \tag{2.6}$$

where $\hat{T}_{\alpha_0}$ and $\hat{T}_{\delta_0}$ are unit vectors tangent at A to the celestial sphere along the parallel of declination δ_0 and the meridian of right ascension α_0 respectively. Thus $(\hat{X}_0, \hat{T}_{\alpha_0}, \hat{T}_{\delta_0})$ represents a right-handed local Cartesian frame at point (α_0, δ_0) and they form an orthonormal basis. See Fig. 3.

The unit vectors $\hat{T}_{\alpha_0}$ and $\hat{T}_{\delta_0}$ may be conveniently obtained from

$$\frac{d\hat{X}_0}{d\alpha_0} = \begin{Bmatrix} -\sin\alpha_0 \\ \cos\alpha_0 \\ 0 \end{Bmatrix} \cos\delta_0 = \hat{T}_{\alpha_0}\cos\delta_0 \tag{2.7}$$

and

$$\hat{T}_{\delta_0} = \frac{d\hat{X}_0}{d\delta_0} = \begin{Bmatrix} -\sin\delta_0\cos\alpha_0 \\ -\sin\delta_0\sin\alpha_0 \\ \cos\delta_0 \end{Bmatrix} . \tag{2.8}$$

It can be seen from Fig. 2 that

$$\sin\psi_0 = \frac{\mu_{\alpha_0}\cos\delta_0}{\mu_0} \tag{2.9}$$

and

$$\cos\psi_0 = \frac{\mu_{\delta_0}}{\mu_0} \tag{2.10}$$

with

$$\mu_0 = [(\mu_{\alpha_0}\cos\delta_0)^2 + \mu_{\delta_0}^2]^{\frac{1}{2}} \; . \tag{2.11}$$

Using the above expressions in (2.6) the value of $\hat{T}_0$ can be given explicitly by

$$\hat{T}_0 = \frac{\mu_{\alpha_0}\cos\delta_0}{\mu_0}\begin{Bmatrix} -\sin\alpha_0 \\ \cos\alpha_0 \\ 0 \end{Bmatrix} + \frac{\mu_{\delta_0}}{\mu_0}\begin{Bmatrix} -\sin\delta_0\cos\alpha_0 \\ -\sin\delta_0\sin\alpha_0 \\ \cos\delta_0 \end{Bmatrix} . \tag{2.12}$$

The velocity vector $\vec{v}$ in equation (2.1) may also be expressed as (see Fig. 1)

$$\vec{v} = R\,\hat{X}_0 + T\,\hat{T}_0 \; . \tag{2.13}$$

The velocity's tangential component at time t_0 can be written as a function of the angular velocity μ_0 and the distance r_0, namely

$$T = \mu_0 r_0 \; . \tag{2.14}$$

After substituting (2.14), (2.13) and (2.4) in (2.1), (2.1) becomes

$$\vec{X}_1 = r_0\,\hat{X}_0 + t(R\,\hat{X}_0 + \mu_0 r_0\,\hat{T}_0) \; , \tag{2.15}$$

or

$$\vec{X}_1 = (r_0 + Rt)\,\hat{X}_0 + \mu_0 r_0 t\,\hat{T}_0 \; . \tag{2.16}$$

Consequently

$$|\vec{X}_1| = [(r_0+Rt)^2 + (\mu_0 r_0 t)^2]^{\frac{1}{2}} = r_1 \; . \tag{2.17}$$

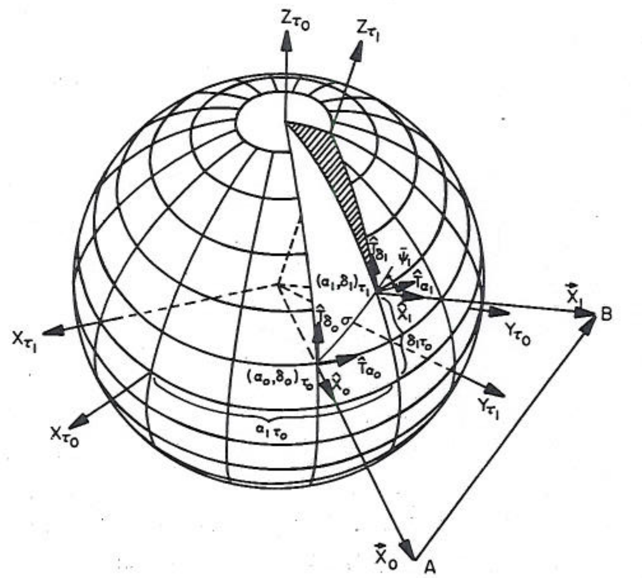

Fig. 3 - Local coordinate systems on the celestial sphere at time of orientation τ_0 and τ_1.

Finally, introducing (2.16) and (2.17) into (2.5) we obtain a rigorous equation for computing $\hat{X}_1$ as a function of $\hat{X}_0$ and other already known or derivable parameters

$$\hat{X}_1 = \frac{r_0+Rt}{[(r_0+Rt)^2+(\mu_0 r_0 t)^2]^{\frac{1}{2}}}\,\hat{X}_0 + \frac{\mu_0 r_0 t}{[(r_0+Rt)^2+(\mu_0 r_0 t)^2]^{\frac{1}{2}}}\,\hat{T}_0 . \tag{2.18}$$

The value of the radial velocity V is generally given in Km/sec measured positively away from the observer. The unit of time $t = t_1 - t_0$ is normally expressed in years. Thus in order to make the units consistent in equation (2.18) we must use for the radial component of $\vec{v}$ at t_0

$$R = V\,n \tag{2.19}$$

where n is the number of seconds per year.

The distance r_0 at t_0 required in equation (2.18) is not directly observed or given in catalogues, but may be computed (in Km) using

$$r_0 \cong \frac{a}{\Pi\rho} \tag{2.20}$$

where a is the semimajor axis of the earth's orbit expressed in Km, Π is the annual parallax of the star expressed in seconds of arc, and ρ is the number of radians per second ($\cong$ sin 1").

3. Comparison with previously published results and final matrix form

Note that equation (2.18) can also be written

$$\hat{X}_1 = \cos\sigma\hat{X}_0 + \sin\sigma\hat{T}_0 \tag{3.1}$$

where σ is the central angle at O between A and B (see Fig. 1).

Substituting (2.6) into (3.1) we have

$$\hat{X}_1 = \cos\sigma\hat{X}_0 + \sin\sigma(\sin\psi_0\hat{T}_{\alpha_0} + \cos\psi_0\hat{T}_{\delta_0}), \tag{3.2}$$

or in matrix form

$$\hat{X}_1 = [\ \hat{X}_0 \ \vdots\ \hat{T}_{\alpha_0} \ \vdots\ \hat{T}_{\delta_0}\ ] \begin{Bmatrix} \cos\sigma \\ \sin\sigma\sin\psi_0 \\ \sin\sigma\cos\psi_0 \end{Bmatrix}$$

$$= \begin{bmatrix} \cos\delta_0\cos\alpha_0 & -\sin\alpha_0 & -\sin\delta_0\cos\alpha_0 \\ \cos\delta_0\sin\alpha_0 & \cos\alpha_0 & -\sin\delta_0\sin\alpha_0 \\ \sin\delta_0 & 0 & \cos\delta_0 \end{bmatrix} \begin{Bmatrix} \cos\sigma \\ \sin\sigma\sin\psi_0 \\ \sin\sigma\cos\psi_0 \end{Bmatrix}. \tag{3.3}$$

This equation can be viewed as a transformation of the components of the unit vector $\hat{X}_1$ given in a coordinate system parallel at O to the local $(\hat{X}_0, \hat{T}_{\alpha_0}, \hat{T}_{\delta_0})$ frame, to the inertial frame (X,Y,Z) at time t_0.

Notice that the matrix of this transformation is just

$$\mathbf{R} = R_3(-\alpha_0)R_2(\delta_0) . \tag{3.4}$$

We may alternatively choose a different ordering for the local triad. For example, if we select a different right-handed local coordinate frame such as $(-\hat{T}_{\delta_0}, \hat{T}_{\alpha_0}, \hat{X}_0)$, equations (3.3) can be rewritten as

$$\hat{X}_1 = [\ -\hat{T}_{\delta_0} \ \vdots\ \hat{T}_{\alpha_0} \ \vdots\ \hat{X}_0\ ] \begin{Bmatrix} -\sin\sigma\cos\psi_0 \\ \sin\sigma\sin\psi_0 \\ \cos\sigma \end{Bmatrix}$$

$$= \begin{bmatrix} \sin\delta_0\cos\alpha_0 & -\sin\alpha_0 & \cos\delta_0\cos\alpha_0 \\ \sin\delta_0\sin\alpha_0 & \cos\alpha_0 & \cos\delta_0\sin\alpha_0 \\ -\cos\delta_0 & 0 & \sin\delta_0 \end{bmatrix} \begin{Bmatrix} -\sin\sigma\cos\psi_0 \\ \sin\sigma\sin\psi_0 \\ \cos\sigma \end{Bmatrix}$$

$$= R_3(-\alpha_0)R_2(\delta_0 - \tfrac{1}{2}\pi) \begin{Bmatrix} -\sin\sigma\cos\psi_0 \\ \sin\sigma\sin\psi_0 \\ \cos\sigma \end{Bmatrix} . \tag{3.5}$$

This is the form presented without proof in [Mueller, 1969, p.115], except that when substitution for the value of σ is made, some simplifications are introduced.

Comparing equations (2.18) and (3.1), it follows that

$$\sin\sigma = \frac{\mu_0 r_0 t}{[(r_0+Rt)^2+(\mu_0 r_0 t)^2]^{\frac{1}{2}}} = \frac{\mu_0 t}{[(1+\nu_0 t)^2+(\mu_0 t)^2]^{\frac{1}{2}}} \tag{3.6}$$

and

$$\cos\sigma = \frac{r_0+Rt}{[(r_0+Rt)^2+(\mu_0 r_0 t)^2]^{\frac{1}{2}}} = \frac{1+\nu_0 t}{[(1+\nu_0 t)^2+(\mu_0 t)^2]^{\frac{1}{2}}} \tag{3.7}$$

where

$$\nu_0 = R/r_0 = Vn\Pi\rho/a . \tag{3.8}$$

Substituting equations (3.6) and (3.7) into (3.1) we obtain

$$\hat{X}_1 = \frac{1}{[(1+\nu_0 t)^2+(\mu_0 t)^2]^{\frac{1}{2}}} \{(1+\nu_0 t)\hat{X}_0+\mu_0 t\hat{T}_0\} \quad (3.9)$$

or making use of (2.12)

$$\hat{X}_1 = \frac{1}{[(1+\nu_0 t)^2+(\mu_0 t)^2]^{\frac{1}{2}}} \{(1+\nu_0 t)\hat{X}_0+\mu_{\alpha_0}\cos\delta_0 t\hat{T}_{\alpha_0}+\mu_{\delta_0} t\hat{T}_{\delta_0}\}. \quad (3.10)$$

Essentially this equation is the one described in the Astronomical Almanac [e.g., 1984, p.B36]. Notice that although the denominator on the right hand side of the equation is the length of the vector it divides, nevertheless it does not depend on the coordinates (α_0,δ_0) of the star at t_0. The only parameters involved in the computation of this length are related to the proper motion of the star itself; consequently stars with the same proper motion, radial velocity and parallax will have the same denominator in (3.10). This fact is not made clear by the normalization process in the Astronomical Almanac, where the whole vector between braces on the right of (3.10) is normalized. Similar numerical normalization is suggested by Green [1985, p.265] to the vector on the right-hand side of equation (3.9) to determine $\hat{X}_1$. Equations (3.9) or (3.10) are rigorous analytical expressions for computing the *intrinsic proper motion* (nutation and precession from t_0 to t_1 are neglected at this time) of stars; however we may look for simpler alternatives.

From (2.17) and (3.8) it immediately follows that

$$r_0/r_1 = [(1+\nu_0 t)^2 + (\mu_0 t)^2]^{-\frac{1}{2}} . \quad (3.11)$$

Therefore equation (3.10) can also be written as

$$\hat{X}_1 = [\ \hat{X}_0 \ \vdots\ \hat{T}_{\alpha_0} \ \vdots\ \hat{T}_{\delta_0}\ ](r_0/r_1)\begin{Bmatrix} 1+t\nu_0 \\ t\mu_{\alpha_0}\cos\delta_0 \\ t\mu_{\delta_0} \end{Bmatrix} \quad (3.12)$$

or recalling (3.3) and (3.4)

$$\hat{X}_1 = \mathbf{R}(r_0/r_1)\begin{Bmatrix} 1+t\nu_0 \\ t\mu_{\alpha_0}\cos\delta_0 \\ t\mu_{\delta_0} \end{Bmatrix} = \mathbf{R}\,\hat{X}_{1\,\ell_0} \quad (3.13)$$

where $\hat{X}_{1\,\ell_0}$ is the unit vector $\hat{X}_1$ expressed on the local coordinate system at time t_0, namely

$$\hat{X}_{1\,\ell_0} = \vec{X}_{1\,\ell_0}/|\vec{X}_{1\,\ell_0}| \quad (3.14)$$

where

$$\vec{X}_{1\,\ell_0} = r_0\begin{Bmatrix} 1+t\nu_0 \\ t\mu_{\alpha_0}\cos\delta_0 \\ t\mu_{\delta_0} \end{Bmatrix} \quad (3.15)$$

and clearly

$$|\vec{X}_{1\,\ell_0}| = |\vec{X}_1| = r_1 . \quad (3.16)$$

Equation (3.13) gives in compact matrix form a conceptually simple but rigorous expression to obtain the effect of intrinsic proper motion on any stellar object. An equivalent matrix expression determined using a completely independent algebraic approach was already given by Eichhorn [1982].

4. Proper motion components at epoch t_1 in terms of the quantities at epoch t_0

In this section we will obtain the angular components of velocity in right ascension $\mu_{\alpha_1}\cos\delta_1$, declination μ_{δ_1} as well as the radial velocity parameter ν_1 $(=\dot{R}_1/r_1)$ at epoch t_1 as a function of the assumed known velocity components $\mu_{\alpha_0}\cos\delta_0$, μ_{δ_0}, and ν_0 $(=\dot{R}/r_0$, eq. (3.8)) at epoch t_0. The problem reduces to the computation of the components of the velocity vector $\vec{v}$ in the local $(\hat{X}_1,\hat{T}_{\alpha_1},\hat{T}_{\delta_1})$ coordinate frame at t_1.

From equation (2.1) or Fig. 1,

$$\vec{v} = (\vec{X}_1-\vec{X}_0)/t . \quad (4.1)$$

In this equation $\vec{v}$, $\vec{X}_1$, and $\vec{X}_0$ are all in the same arbitrary coordinate system. In particular this is taken to be the (X,Y,Z) inertial frame at some epoch. We will need to identify the epoch of this frame, a function of time because of nutation and precession. Thus we write,

$$\vec{v}_{\tau_1} = (\vec{X}_{1\,\tau_1}-\vec{X}_{0\,\tau_1})/t \ ; \quad \vec{v}_{\tau_0} = (\vec{X}_{1\,\tau_0}-\vec{X}_{0\,\tau_0})/t . \quad (4.2)$$

The subscript τ_i's identify the *epoch of orientation* of the inertial coordinate system in which the component of the vector is expressed. Incidentally we must also have a subindex in order to properly identify the epoch of the system to which the curvilinear (equatorial) coordinates are referred, e.g., $(\alpha_0,\delta_0)_{\tau_0}$, $(\alpha_1,\delta_1)_{\tau_0}$, $(\alpha_1,\delta_1)_{\tau_1}$, etc. The subindices will be omitted when they are clear by the context.

Denote by $\vec{v}_{\ell_0}$ and $\vec{v}_{\ell_1}$ the velocity vector (components) expressed in the local $(\hat{X}_0,\hat{T}_{\alpha_0},\hat{T}_{\delta_0})_{\tau_0}$ and $(\hat{X}_1,\hat{T}_{\alpha_1},\hat{T}_{\delta_1})_{\tau_1}$ systems at *epoch of place* t_0 and t_1 respectively but defined through the curvilinear lines (hour circles and circles of declination) of the inertial systems at τ_0 and τ_1 (see Fig. 3). Notice that in general $t_i \neq \tau_i$ for i = 0,1. Then the velocity components in radians, or as proportional parts of the distance, are given by

$$\begin{Bmatrix} \nu_1 \\ \mu_{\alpha_1}\cos\delta_1 \\ \mu_{\delta_1} \end{Bmatrix} = \vec{v}_{\ell_1}/r_1 \quad \text{and} \quad \begin{Bmatrix} \nu_0 \\ \mu_{\alpha_0}\cos\delta_0 \\ \mu_{\delta_0} \end{Bmatrix} = \vec{v}_{\ell_0}/r_0 . \quad (4.3)$$

Also, (compare equations (3.3) and (3.4)),

$$\vec{v}_{\ell_0} = R_2(-\delta_0)R_3(\alpha_0)\ \vec{v}_{\tau_0} \ ; \ \vec{v}_{\ell_1} = R_2(-\delta_1)R_3(\alpha_1)\ \vec{v}_{\tau_1} . \quad (4.4)$$

Since

$$\vec{X}_{0_{\tau_1}} = [N][P]\,\vec{X}_{0_{\tau_0}}\ ,\quad \vec{X}_{1_{\tau_1}} = [N][P]\,\vec{X}_{1_{\tau_0}} \tag{4.5}$$

where [P] and [N] are respectively the orthogonal precession (from τ_0 to τ_1) and nutation (at τ_1) matrices as given for example in [Mueller, 1969, p.65 and 75]. Recall that [N] is the unit matrix when transforming between mean coordinate systems or places.

As expected it follows from (4.2) and (4.5) that

$$\vec{v}_{\tau_1} = [N][P]\,\vec{v}_{\tau_0} \tag{4.6}$$

and thus from (4.3) and (4.4), that

$$\begin{Bmatrix} \nu_1 \\ \mu_{\alpha_1}\cos\delta_1 \\ \mu_{\delta_1} \end{Bmatrix} = (r_0/r_1)[R] \begin{Bmatrix} \nu_0 \\ \mu_{\alpha_0}\cos\delta_0 \\ \mu_{\delta_0} \end{Bmatrix} \tag{4.7}$$

where $[R]$ is a convenient short designation for an orthogonal matrix given by

$$[R] = R_2(-\delta_1)R_3(\alpha_1)[N][P]R_3(-\alpha_0)R_2(\delta_0). \tag{4.8}$$

Equation (4.7) solves the stated problem of this section in principle, since from (3.11) r_0/r_1 is known and $\hat{X}_{1_{\tau_1}}$ from equation (3.13) and equation (3.4) yields $(\alpha_1,\delta_1)_{\tau_1}$ needed to compute $[R]$. That is, α_1 and δ_1 at t_1 with respect to the frame $(X,Y,Z)_{\tau_1}$ are found from

$$\hat{X}_{1_{\tau_1}} = \begin{Bmatrix} \cos\delta_1\cos\alpha_1 \\ \cos\delta_1\sin\alpha_1 \\ \sin\delta_1 \end{Bmatrix} = (r_0/r_1)[N][P]R_{\kappa} \begin{Bmatrix} 1+t\nu_0 \\ t\mu_{\alpha_0}\cos\delta_0 \\ t\mu_{\delta_0} \end{Bmatrix}. \tag{4.9}$$

Equations (4.7) and (4.8) in conjunction with (4.9) define a rigorous straightforward updating of proper motion components that has much to recommend it for actual numerical computations. Recall that the above equations involve epochs of orientation τ_i, i=1,0 and place t_i, i=1,0. In a sense they are more general that other matrix expressions recently introduced assuming $t_i=\tau_i$ and involving transformations between mean epochs. Consult [Standish, 1982; Aoki et al., 1983; Astronomical Almanac, 1984, p.X]. Notice that equation (4.7) is the transformation of the components of proper motion with respect to the local coordinate system at $(\alpha_0,\delta_0)_{\tau_0}$ to the local coordinate system with origin at $(\alpha_1,\delta_1)_{\tau_1}$. This transformation is easily visualized through rotation matrices assuming all coordinate systems located at the origin of the celestial sphere. (See Fig. 3). The matrix of the transformation is precisely equation (4.8).

Noticing that (see the left-hand equality in equation (4.9); a similar equation holds for $\hat{X}_{0_{\tau_0}}$)

$$\hat{X}_{1_{\tau_1}} = R_3(-\alpha_1)R_2(\delta_1)\begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \tag{4.10}$$

equation (4.9) can be written

$$\begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} = (r_0/r_1)[R]\begin{Bmatrix} 1+t\nu_0 \\ t\mu_{\alpha_0}\cos\delta_0 \\ t\mu_{\delta_0} \end{Bmatrix}. \tag{4.11}$$

Observe that taking the norm of both sides of (4.11), using equation (2.11), yields an alternative derivation of (3.11). Furthermore, denoting the first column of $[R]^t$ by $\vec{\rho}_1$, from (4.11), multiplying both sides by $[R]^t$, we find

$$\vec{\rho}_1 = (r_0/r_1)\begin{Bmatrix} 1+t\nu_0 \\ t\mu_{\alpha_0}\cos\delta_0 \\ t\mu_{\delta_0} \end{Bmatrix} = \hat{X}_{1_{\ell_0}}. \tag{4.12}$$

The first row of $[R]$ is the vector $(\vec{\rho}_1)^t$. Using this fact in equation (4.7) allows us to solve for the first (radial) component ν_1

$$\nu_1 = (r_0/r_1)^2\begin{Bmatrix} 1+t\nu_0 \\ t\mu_{\alpha_0}\cos\delta_0 \\ t\mu_{\delta_0} \end{Bmatrix}^t \begin{Bmatrix} \nu_0 \\ \mu_{\alpha_0}\cos\delta_0 \\ \mu_{\delta_0} \end{Bmatrix}. \tag{4.13}$$

Expanding and using (2.11) and (3.11), this becomes

$$\nu_1 = [\nu_0 + (\nu_0^2+\mu_0^2)t]/[(1+\nu_0 t)^2 + \mu_0^2 t^2]. \tag{4.14}$$

This equation was independently derived in [Eichhorn and Rust, 1970] or [Eichhorn, 1974, p.84].

From (4.7) and the orthogonality of $[R]$

$$\nu_1^2 + \mu_1^2 = (\nu_0^2+\mu_0^2)/[(1+\nu_0 t)^2 + \mu_0^2 t^2]. \tag{4.15}$$

Substituting for ν_1 from (4.14) yields

$$\mu_1 = \mu_0/[(1+\nu_0 t)^2 + \mu_0^2 t^2]\ . \tag{4.16}$$

Equations (4.14), (4.15) and (4.16) hold without neglect of precession and nutation. It comes as no surprise to find that ν_1 and μ_1 are not affected by the change in orientation of the local coordinate system at τ_0 introduced by [N][P]. The component ν_1 is measured along the normal to the celestial sphere, and therefore is invariant with respect to a rotation of coordinates on the sphere. μ_1 is the magnitude of the tangential velocity vector, which is independent of the particular local coordinate system in which the tangent plane coordinates of this vector are expressed.

The remaining two components at t_1, the projections of the velocity vector on the unit tangent vectors along the $(\alpha_1,\delta_1)_{\tau_1}$ lines, $\mu_{\alpha_1}\cos\delta_1$ and μ_{δ_1}, are dependent on the inertial coordinate frame in which α_1 and δ_1 are expressed, hence dependent on $[N(\tau_1)]$ and $[P(\tau_0,\tau_1)]$.

Explicit values of α_1 and δ_1 at τ_0 were given in [Eichhorn and Rust, 1970] and [Eichhorn, 1974, p.22]. They may be computed using equations (2.18) or (3.13);

also from (4.9) after assuming [N][P]=I. For the sake of completeness these values are written below

$$\tan\delta_{1_{\tau_0}} = [(1+\nu_0 t)\tan\delta_0+\mu_{\delta_0} t]/(A^2+\mu^2_{\alpha_0}t^2)^{\frac{1}{2}} \tag{4.17}$$

$$\tan(\alpha_{1_{\tau_0}}-\alpha_0) = \mu_{\alpha_0}t/A \tag{4.18}$$

where

$$A=1+\nu_0 t-\mu_{\delta_0}\tan\delta_0 t\ . \tag{4.19}$$

To clarify this point, let's obtain the components of proper motion at t_1 and τ_1 from the components of proper motion at t_1 and τ_0 as follows

$$\begin{Bmatrix} \nu_1 \\ \mu_{\alpha_1}\cos\delta_1 \\ \mu_{\delta_1} \end{Bmatrix}_{\tau_1} = (r_0/r_1)[R]\mathbf{R}^t_{\bullet}R_3(-\alpha_{1_{\tau_0}})R_2(\delta_{1_{\tau_0}})\begin{Bmatrix} \nu_1 \\ \mu_{\alpha_1}\cos\delta_1 \\ \mu_{\delta_1} \end{Bmatrix}_{\tau_0}. \tag{4.20}$$

Notice that from (3.4) and (4.8)

$$[R]\mathbf{R}^t_{\bullet} = R_2(-\delta_1)R_3(\alpha_1)[N][P]\ .$$

While the values $(\alpha_1,\delta_1)_{\tau_0}$ above are given by equations (4.17) to (4.19) the equatorial coordinates (α_1,δ_1) refer to the inertial frame at τ_1 and should be computed from equation (4.9). The subindices τ_0 and τ_1 attached to the proper motion components in equation (4.20) are needed to stress the difference in epoch of orientation although they refer to the same epoch of observation.

From equation (4.11) it easily follows that

$$\begin{Bmatrix} \nu_0 \\ \mu_{\alpha_0}\cos\delta_0 \\ \mu_{\delta_0} \end{Bmatrix} = (1/t)\left\langle (r_1/r_0)[R]^t\begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}\right\rangle \tag{4.21}$$

and introducing the above in equation (4.7) we get

$$\begin{Bmatrix} \nu_1 \\ \mu_{\alpha_1}\cos\delta_1 \\ \mu_{\delta_1} \end{Bmatrix} = (1/t)\left\langle \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} - (r_0/r_1)[R]\mathbf{R}^t_{\bullet}\ \hat{X}_{0_{\tau_0}}\right\rangle. \tag{4.22}$$

Inserting (4.11) into (4.22) we arrive at

$$\begin{Bmatrix} \nu_1 \\ \mu_{\alpha_1}\cos\delta_1 \\ \mu_{\delta_1} \end{Bmatrix} = (1/t)[R]\mathbf{R}^t_{\bullet}\ \{\hat{X}_1 - (r_0/r_1)\hat{X}_0\}_{\tau_0}\ . \tag{4.23}$$

Finally, replacing above the value of $\hat{X}_1$ from our basic equation (3.9) we get

$$\begin{Bmatrix} \nu_1 \\ \mu_{\alpha_1}\cos\delta_1 \\ \mu_{\delta_1} \end{Bmatrix} = (r_0/r_1)[R]\mathbf{R}^t_{\bullet}\ \{\nu_0\hat{X}_0 + \mu_0\hat{T}_0\}\ . \tag{4.24}$$

This expression will rigorously give the components of proper motion at epoch of place t_1 with respect to the new $(X,Y,Z)_{\tau_1}$ inertial coordinate frame at epoch of orientation τ_1 as a function of known quantities at the epoch of place t_0 and epoch of orientation τ_0. Equation (4.24) is a different form of (4.7) and as before the values (α_1,δ_1) should be computed from equation (4.9). Notice that the approach is characterized by its mathematical simplicity, based on well known quantities discussed in previous sections. This remedies an omission in standard references, which either give an algorithm without proof, or sketch derivations that from a didactic point of view are considerably less direct and compact than the ones presented here.

Let's turn our attention to the computation of proper motion of a star at epoch t_0 from its positions (with respect to the same inertial frame) and distances at epochs t_0 and t_1.

Actually, the final relationship can be obtained from equation (4.21) which may be written in the form

$$\begin{Bmatrix} \nu_0 \\ \mu_{\alpha_0}\cos\delta_0 \\ \mu_{\delta_0} \end{Bmatrix} = (1/t)\left\langle (r_1/r_0)\ \mathbf{R}^t_{\bullet}\hat{X}_{1_{\tau_0}} - \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}\right\rangle \tag{4.25}$$

where

$$\mathbf{R}^t_{\bullet}\hat{X}_{1_{\tau_0}} = \begin{Bmatrix} \cos\delta_0\cos\delta_1\cos(\alpha_1-\alpha_0)+\sin\delta_1\sin\delta_0 \\ \cos\delta_1\sin(\alpha_1-\alpha_0) \\ -\sin\delta_0\cos\delta_1\cos(\alpha_1-\alpha_0)+\cos\delta_0\sin\delta_1 \end{Bmatrix}. \tag{4.26}$$

Simple inspection of equation (4.25) shows that we can eliminate the distance dependence on the proper motion components by dividing the second and third equations by the first, leaving finally

$$\begin{Bmatrix} \mu_{\alpha_0}\cos\delta_0 \\ \mu_{\delta_0} \end{Bmatrix} = \frac{1+\nu_0 t}{t[\cos\delta_0\cos\delta_1\cos(\alpha_1-\alpha_0)+\sin\delta_1\sin\delta_0]}\times \begin{Bmatrix} \cos\delta_1\sin(\alpha_1-\alpha_0) \\ -\sin\delta_0\cos\delta_1\cos(\alpha_1-\alpha_0)+\cos\delta_0\sin\delta_1 \end{Bmatrix}. \tag{4.27}$$

Notice that this is a close form for computing the proper motion components at t_0 and is dependent only on the positions of the star at t_0 and t_1 and the radial velocity of the star at t_0 (recall that $\nu_0=R/r_0$ where r_0 can now be equal to 1). See also [Eichhorn, 1982; equation (38)]. It should be pointed out here that the two components of proper motion in equation (4.27) after multiplying by t reduce to the *standard coordinates* (ξ,η) [e.g., Bomford, 1980, p.549; Green, 1985, p.321] when $\nu_0=0$.

Finally an alternative equation to (4.7) or (4.24) completely independent of precession and nutation will be discussed. It is clear by simple inspection of Fig. 3 that the rotation matrix $[\mathcal{R}]$ given by (4.8) can be replaced by

$$[R_a] = R_1(-\bar{\psi}_1)R_3(\sigma)R_1(\bar{\psi}_0) \qquad (4.28)$$

where $\bar{\psi}_i=\frac{1}{2}\pi-\psi_i$, i=0,1. To prove this analytically (even when $[N][P]\neq I$), it will be sufficient to show that $[R_a]$ satisfies equation (4.11). If this is so, then we have

$$[\mathcal{R}]\vec{m} = [R_a]\vec{m} \quad \text{for any vector} \quad \vec{m} = \begin{Bmatrix} \nu_0 \\ \mu_{\alpha_0}\cos\delta_0 \\ \mu_{\delta_0} \end{Bmatrix}.$$

By choosing three such vectors, linearly independent, so that the inverse of the matrix $[\vec{m}_1 \vdots \vec{m}_2 \vdots \vec{m}_3]$ exists, it follows

$$[\mathcal{R}]=[R_a]. \qquad (4.29)$$

Upon expansion, the first column of $[R_a]^t$ is found not to depend on ψ_1. Since $[R_a]$ is required to satisfy (4.11), (4.12) becomes

$$\vec{\rho}_{1_a} = \begin{Bmatrix} \cos\sigma \\ \sin\sigma\sin\psi_0 \\ \sin\sigma\cos\psi_0 \end{Bmatrix} = (r_0/r_1)\begin{Bmatrix} 1+t\nu_0 \\ t\mu_{\alpha_0}\cos\delta_0 \\ t\mu_{\delta_0} \end{Bmatrix}. \qquad (4.30)$$

These equations when solved for $\sin\sigma$, $\cos\sigma$, $\sin\psi_0$, $\cos\psi_0$, give equations (3.6), (3.7), (2.9) and (2.10), respectively, thus confirming the identity of σ and ψ_0 with the angles previously so labeled. Incidentally, notice that (4.30) into (3.3) gives equation (3.13). After substituting from (2.9), (2.10) for $\sin\psi_0$ and $\cos\psi_0$ in (4.28), then the resulting $[R_a]$ into (4.7), this equation becomes

$$\begin{Bmatrix} \nu_1 \\ \mu_{\alpha_1}\cos\delta_1 \\ \mu_{\delta_1} \end{Bmatrix} = (r_0/r_1)R_1(-\bar{\psi}_1)R_3(\sigma)\begin{Bmatrix} \nu_0 \\ \mu_0 \\ 0 \end{Bmatrix}. \qquad (4.31)$$

Equation (4.31) may have practical advantages in some astrometric applications where it is possible to measure more precisely the angles $\bar{\psi}_1$ and σ than the coordinates $(\alpha_1,\delta_1)_{\tau_1}$.

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References

Aoki, S., Soma, M., Kinoshita, H., and Inoue, K., 1983. Conversion matrix of epoch B 1950.0 FK 4-based positions of stars to epoch J 2000.0 positions in accordance with the new IAU resolutions, *Astron. Astrophys.*, *128*, 263-267.

Astronomical Almanac, Annual publication. U.S. Government Printing Office, Washington, D.C., and Her Majesty's Stationery Office, London.

Bomford, G., 1980. *Geodesy*, 4th edition, Clarendon Press, Oxford.

Eichhorn, H., 1974. *Astronomy of Star Positions*, Ungar Pub. Co., New York, NY.

Eichhorn, H., 1982. Strenge Formeln für das Rechnen mit Eigenbewegungen, Sitz. Ber. d. Öst. Akad. Wiss., math.-naturw. Kl. Abt. II *191*, 439-459.

Eichhorn, H. and Rust, A., 1970. Rigorous computation of proper motions and their effects on star positions. *Astron. Nachr.*, *292*, 37-38.

Green, R.E., 1985. *Spherical Astronomy*, Cambridge Univ. Press, New York, NY.

Kustaanheimo, P., 1960. Time derivatives of the components of proper motions of stars, *Astron. J.*, *65*, 46-47.

Mueller, I.I., 1969. *Spherical and Practical Astronomy as Applied to Geodesy*, Frederic Ungar Pub. Co., New York, N.Y.

Murray, C.A., 1983. *Vectorial Astrometry*, A. Hilger, Bristol, England.

Schlesinger, F., 1917. On the secular changes in the proper-motions and other elements of certain stars, *Astron. J.*, *30*, 137-138.

Scott, F.P. and Hughes, J.A., 1964. Computation of apparent places for the Southern Reference Star Program, *Astron. J.*, *69*, 368-371.

Smart, W.M., 1956. *Spherical Astronomy*, Cambridge University Press, New York, NY.

Standish, E.M., 1982. Conversion of positions and proper motions from B1950.0 to the IAU System at J2000.0, *Astron. Astrophys.*, *115*, 20-22.

Stumpff, P., 1985. Rigorous treatment of the heliocentric motion of stars, *Astron. Astrophys.*, *144*, 232-240.

Taff, L.G., 1981. *Computational Spherical Astronomy*, John Wiley & Sons, New York, NY.

Woolard, E.W., and Clemence, G.M., 1966. *Spherical Astronomy*, Academic Press, New York, NY.